TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 12, December 1995

PERIODIC ORBITS OF n-BODY TYPE PROBLEMS: THE FIXED PERIOD CASE

HASNA RIAHI

ABSTRACT. This paper gives a proof of the existence and multiplicity of periodic solutions to Hamiltonian systems of the form

$$\text{(A)} \qquad \begin{cases} m_i\ddot{q}_i + \frac{\partial V}{\partial q_i}(t, q) = 0 \\ q(t+T) = q(t), \ \forall t \in \Re. \end{cases}$$

where $q_i \in \Re^\ell$, $\ell \geq 3$, $1 \leq i \leq n$, $q = (q_1, \dots, q_n)$ and

$$V = \sum_{1 \leq i \neq j}^{n} V_{ij}(t, q_i - q_j)$$

with $V_{ij}(t, \xi)$ T-periodic in t and singular in ξ at $\xi = 0$. Under additional hypotheses on V, when (A) is posed as a variational problem, the corresponding functional, I, is shown to have an unbounded sequence of critical values if the singularity of V at 0 is strong enough. The critical points of I are classical T-periodic solutions of (A). Then, assuming that I has only non-degenerate critical points, up to translations, Morse type inequalities are proved and used to show that the number of critical points with a fixed Morse index k grows exponentially with k, at least when $k \equiv 0, 1 \pmod{\ell - 2}$. The proof is based on the study of the critical points at infinity done by the author in a previous paper and generalizes the topological arguments used by A. Bahri and P. Rabinowitz in their study of the 3-body problem. It uses a recent result of E. Fadell and S. Husseini on the homology of free loop spaces on configuration spaces. The detailed proof is given for the 4-body problem then generalized to the n-body problem.

1. INTRODUCTION

The aim of this paper is to prove the existence and multiplicity of periodic solutions with fixed period to Hamiltonian systems of the form

$$\text{(1.1)} \qquad \begin{cases} m_i\ddot{q}_i + \frac{\partial V}{\partial q_i}(t, q) = 0 \\ q(t+T) = q(t), \ \forall t \in \Re, \end{cases}$$

where $q_i \in \Re^\ell$, $\ell \geq 3$, $1 \leq i \leq n$, $q = (q_1, \dots, q_n)$ and $V : \Re \times F_n(\Re^\ell) \longrightarrow \Re$.

Received by the editors November 11, 1993.
1991 *Mathematics Subject Classification*. Primary 58F22; Secondary 70F10, 70H05.
Key words and phrases. n-body problems, periodic solutions, generalized Morse inequalities.

Here $F_n(\Re^\ell)$ denotes the n^{th} configuration space on $\Re^\ell$

$$F_n(\Re^\ell) = \{(q_1, \dots, q_n) \in (\Re^\ell)^n / \ q_i \neq q_j \text{ if } i \neq j\}.$$

Since the arguments used here are valid for any choice of $m_i > 0$, $1 \le i \le n$, one can take $m_i = 1$, $1 \le i \le n$ and write (1.1) as

$$\text{(HS)} \qquad \begin{cases} \ddot{q} + \nabla_q V(t, q) = 0 \\ q(t+T) = q(t), \ \forall \in \Re. \end{cases}$$

V is given by

$$V = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} V_{ij}(t, q_i - q_j) \tag{1.2}$$

where for each i, j the function V_{ij} is T-periodic in t and satisfies

(V_1) $V_{ij}(t, \xi) \in \mathscr{C}^2(\Re \times (\Re^\ell - \{0\}), \Re)$,

(V_2) $V_{ij}(t, \xi) < 0$,

(V_3) $V_{ij}(t, \xi) \longrightarrow 0$ and $V'_{ij}(t, \xi) \longrightarrow 0$ as $|\xi| \longrightarrow +\infty$,

(V_4) $V_{ij}(t, \xi) \longrightarrow -\infty$ as $\xi \longrightarrow 0$,

(V_5) For all $M > 0$, there is an $R > 0$ such that $|\xi| > R$ implies that $\nabla_\xi V_{ij}(t, \xi).\xi > M|\nabla_\xi V_{ij}(t, \xi)|$,

(V_6) There exists $U_{ij} \in \mathscr{C}^1(\Re \times (\Re^\ell - \{0\}), \Re)$ such that $U_{ij}(t, \xi) \longrightarrow \infty$ as $\xi \longrightarrow 0$ and $-V_{ij}(t, \xi) \ge |\nabla_\xi U_{ij}(t, \xi)|^2$.

Potentials having the following form

$$V(q) = -\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\alpha_{ij}}{|q_i - q_j|^{\beta_{ij}}} \tag{1.3}$$

where α_{ij} and β_{ij} are positive, satisfy (V_1)–(V_5). Moreover (V_6) is satisfied if and only if $\beta_{ij} \ge 2$ for all i, j. Hence the classical n-body problem, where $\beta_{ij} = 1$, $1 \le i \neq j \le n$, does not satisfy the strong force hypothesis (V_6).

Once $T > 0$ is chosen, (HS) is posed as a variational problem in order to find T-periodic solutions.

Let $E = W_T^{1,2}(\Re, (\Re^\ell)^n)$ be equipped with the norm

$$\|q\| = \left(\int_0^T |\dot{q}|^2 \, dt + [q]^2\right)^{\frac{1}{2}}$$

where

$$[q] = \frac{1}{T}\int_0^T q(s)\, ds.$$

The functional corresponding to (HS) is

$$I(q) = \int_0^T \left(\frac{1}{2}|\dot{q}|^2 - V(t, q)\right) dt.$$

(V_6) implies that if $q \in E$ and $I(q) < \infty$, then $q \in \Lambda_n$ where

$$\Lambda_n = \{q \in E / \ q_i(t) \neq q_j(t) \text{ for all } i \neq j \text{ and } t \in [0, T]\}.$$

Critical points of I in Λ_n are then easily seen to be classical T-periodic solutions of (HS). The main result is

Theorem 1. *If V is autonomous and satisfies $(V_1)-(V_6)$, then for each $T>0$, I possesses an unbounded sequence of critical values.*

Since, in the proof, no explicit use is made of the fact that V is independent of t, we also get the following result

Theorem 1′. *Suppose $V=V(t,q)\colon \Re\times F_n(\Re^\ell)\longrightarrow\Re$ is T-periodic in t and satisfies $(V_1)-(V_6)$. Then the functional I has an unbounded sequence of critical values which correspond to T-periodic solutions of* (HS).

If one assumes that I has only non-degenerate critical points, modulo translations, then we also have the following theorem which shows that the number of periodic orbits increases exponentially with the Morse index.

Theorem 2. *Let V satisfy $(V_1)-(V_6)$. Assume that if $I'(q)=0$ then $\overline{m}(q)-m(q)=\ell$ where $m(q)$ is the Morse index of q and $\overline{m}(q)$ is the generalized Morse index of q. Let N_k denote the number of critical points, q, of I such that $m(q)=k$. Then,*

$$N_k \geq c_n\,\frac{(n-1)^r}{r(r-1)}$$

if k is large enough and $k=r(l-2)$ or $k=r(l-2)+1$. Here, c_n is a positive constant which depends on n and satisfies $0<c_n<1$.

Different papers have dealt with n-body type problems under different conditions. H. Poincaré [18] studied the 3-body problem in a very particular case using the convergence of some series. A. Bahri and P. Rabinowitz studied 2-body type problems in [5] and 3-body type problems in [6] under the same conditions as the ones considered here. For 2-body type problems, we also refer to the works by A. Ambrosetti and V. Coti-Zelati [1, 2], M. Degiovanni and F. Giannoni [8], W. Gordon [11], C. Greco [12] and the references therein. Concerning n-body type problems, V. Coti-Zelati studied the symmetrical case ($V_{ij}(\xi)=V_{ji}(\xi)$ or more generally, $V(t+\frac{T}{2})-q)=V(t,q))$ in [7] and recent work by P. Majer and S. Terracini [16, 17], using conditions similar to the ones assumed here but slightly different at infinity, proves the existence of an infinite number of solutions for the n-body problem with the strong force hypothesis. It relies on a variant of Ljusternik-Schnirelman variational theory and on a collision index related to the relative category.

The proof given here, uses the description given in [19] of the critical points at infinity arising from the failure of the Palais-Smale condition and of the topology of their unstable manifolds. It gives us a more complete account of the solutions at least when the functional I is assumed to have only non-degenerate critical points which, by [15] and [3, 4], can always be assumed to be the case if we perturb the potential a little bit.

To prove Theorems 1, 1′ and 2, we use the results of [19] where it was proved that if we approximate I by a new functional $\widetilde{I}$ with non degenerate critical points, then $\widetilde{I}^M=\{q\in E\ /\ \widetilde{I}(q)\leq M\}$ has the same homology as $\widetilde{I}^{\epsilon_1}\cup D_M\cup D_M^\infty$ where D_M is the union of the unstable manifolds of all classical critical points of $\widetilde{I}$ in $\widetilde{I}^M-\widetilde{I}^{\epsilon_1}$ and D_M^∞ is a similar set for the critical points at infinity. In Section 2, taking $n = 4$ and arguing by contradiction, I is shown to have an unbounded sequence of critical values. Indeed, if the set of critical values of I were bounded then, for k large enough, the following inequality would be true:

$$\operatorname{rank} H_k(\Lambda_4) \leq 4(\operatorname{rank} H_{k-(l-1)}(\Lambda_3) + \operatorname{rank} H_k(\Lambda_3)) + ak + b$$

where a and b are positive constants. On the other hand, a recent result of E. Fadell and S. Husseini [10], shows that the above inequality is false for k large enough. This contradiction establishes the result for the 4-body problem. In Section 3, assuming that I has only non-degenerate critical points, up to translations, Morse type inequalities are proved for the 4-body problem. They give us estimates on the number of critical points with a fixed Morse index k, which is, then, seen to grow exponentially with k if $k \equiv 0, 1 \pmod{l-2}$. In Section 4, these results are generalized to the n-body problem for $n \geq 4$, using the results of [19] and the following theorem of E. Fadell and S. Husseini [10]:

Theorem (E. Fadell and S. Husseini). *If Y is a topological space, let $\Omega(Y)$ be the free loop space of Y. Let $X = \{x_s\}$ denote a fixed set of mutually distinct points in $\Re^\ell$ and $X_j = \{x_1, \dots, x_j\}$. Then, using coefficients in $\mathscr{Z}_2$,*

$$H_*(\Lambda_n) = H_*(\Omega(\Re^\ell - X_{n-1})) \otimes H_*(\Omega(\Re^\ell - X_{n-2})) \otimes \dots \otimes H_*(\Omega(\Re^\ell - X_1)).$$

Since, by results of M. Vigué-Poirrier [20], E. Fadell and S. Husseini [10], it is known that

$$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j); \mathscr{Z}_2) \neq 0, \text{ only if } k \equiv 0, 1 \pmod{l-2},$$

$$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j); \mathscr{Z}_2) \leq j^r \text{ for } k = r(l-2) \text{ and } k = r(l-2)+1$$

and

$$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j); \mathscr{Z}_2) \geq \frac{j^r}{r(r-1)}(1 - rj^{-\frac{\ell}{2}}) \text{ for } k = r(l-2)$$
$$\text{and } k = r(l-2)+1,$$

this yields the results for the n-body problem associated with (HS) when the singularity of V at 0 satisfies the strong force hypothesis, (V_6), which eliminates collision orbits.

2. Finding periodic orbits for the 4-body problem

Theorem 2.1. *If V satisfies $(V_1) - (V_6)$, then for each $T > 0$, I possesses an unbounded sequence of critical values.*

Sketch of the proof. Suppose the set of critical values of I is bounded by a. Let $M > a$. Let $\widetilde{I}$ be an approximating functional for I with non-degenerate critical points defined as in [19] and let $\widetilde{Z}$ be the pseudogradient vector field constructed for $\widetilde{I}$ in [19]. It was shown in [19], following [6], that if ϵ_1 is small enough, then I has no critical points under the level $2\epsilon_1$ and any trajectory

$q(\tau)$ of

$$\frac{dq}{d\tau} = -\widetilde{Z}(q)$$

with $q(0) \in \widetilde{I}^{M+1} = I^{M+1}$ which does not enter $\widetilde{I}^{\epsilon_1} = I^{\epsilon_1}$ or does not converge to a critical point of $\widetilde{I}$ has a limit. The set of such limits, $\mathscr{H}$, is called the set of critical points at infinity of $\widetilde{I}$. An unstable manifold is associated to each critical point at infinity and the union of these unstable manifolds is denoted by $\mathscr{D}^{\infty}_{M+1}$. Let $\mathscr{D}_{M+1}$ be the union of the unstable manifolds of the critical points of $\widetilde{I}$ in $\widetilde{I}^{M+1}$. Then, in [19] it was shown that $\mathscr{D}_{M+1}$ is an Euclidian neighborhood retract (E.N.R.) of dimension m. The following sets were also defined in [19]: $\mathscr{W}^{\infty}$, a set with a piecewise smooth boundary which contains $I^{\epsilon_1} \cup \mathscr{D}^{\infty}_{M+1}$ in its interior and $\mathscr{V}_\epsilon = \mathscr{V}_\epsilon(\mathscr{D}_{M+1})$, an ϵ-neighborhood of $\mathscr{D}_{M+1}$. $\mathscr{W}^{\infty}$ is chosen so that $\widetilde{I}_{M+1} = I_{M+1}$ retracts by deformation onto $\mathscr{W}^{\infty} \cup \mathscr{V}_\epsilon$ and $\mathscr{W}^{\infty}$ has the homotopy type of $I^{\epsilon_1} \cup \mathscr{D}^{\infty}_{M+1}$. The sets $\mathscr{V}_\epsilon$ and $\mathscr{V}_\epsilon \cap \mathscr{W}^{\infty}$ are absolute neighborhood retracts (A.N.R.'s) and their homologies vanish in dimension larger than $m+1$. Using these results, we will show that the Betti numbers of $\mathscr{W}^{\infty} \cup \mathscr{V}_\epsilon$ in $\mathscr{Z}_2$ homology, hence the Betti numbers of Λ_4, must satisfy the following inequality, for k large enough

$$\operatorname{rank} H_k(\Lambda_4) \leq 4(\operatorname{rank} H_{k-(l-1)}(\Lambda_3) + \operatorname{rank} H_k(\Lambda_3)) + ak + b \tag{2.1}$$

where a and b are positive constants.

On the other hand, by a result of S. Husseini and E. Fadell [10], for k large enough, the Betti numbers of Λ in $\mathscr{Z}_2$ homology satisfy the following inequality

$$\operatorname{rank} H_k(\Lambda_4) > f(k) + g(k) \operatorname{rank} H_k(\Lambda_3) \tag{2.2}$$

for any polynomials f and g.

This gives us a contradiction which establishes Theorem 2.1.

Detailed proof. First, we want some estimates for the generalized Morse indices of the critical points of $\widetilde{I}$ in I^{M+1}. If we suppose that the critical points of I lie in I^a, then by [19] (Proposition 2.1), there exists a $\delta(a)$ such that for any critical point q of I in I^a, we have

$$\inf_{\substack{i \neq j \\ t \in [0,1]}} |q_i(t) - q_j(t)| \geq \delta(a).$$

Moreover, by [19] (Corollary 3.9), these critical points of I are uniformly bounded by $c(\epsilon_1, a)$ in $\widetilde{\Lambda}$, the quotient space of $\Lambda = \Lambda_4$ by the translations of $\Re^{\ell}$. For any such q and any $\varphi \in E$,

$$I''(q)(\varphi, \varphi) = \int_0^1 [|\dot{\varphi}|^2 - \sum_{i \neq j} V''_{ij}(q_i - q_j)(\varphi_i - \varphi_j).(\varphi_i - \varphi_j)] \tag{2.3}$$

The generalized Morse index of q is the dimension of the subspace of E on which $I''(q)$ is non-positive definite. The form of I'' and above remarks show that the generalized Morse index of any critical point of I in I^a is bounded above by a positive integer $m = m(a)$. We showed, in [19] (Section 4), that

$\|I - \tilde{I}\|_{\mathscr{C}^2}$ can be made as small as we want and critical points of $\tilde{I}$ lie in a small neighborhood of those of I. Hence, m is also an upper bound for the generalized Morse index of any critical point of $\tilde{I}$. It can be shown that $\mathscr{D}_{M+1}$ is an E.N.R. of dimension at most m. In the sequel, m is the dimension of $\mathscr{D}_{M+1}$.

Now, we choose k such that

$$k \geq \max(m+1, 5\ell). \tag{2.4}$$

Let

$$\{z\} \in H_k(\Lambda, \mathscr{Z}_2).$$

Then, $\{z\}$ may be represented by a chain z having support in a compact set $K \in \Lambda$.

Choose $M > a$ such that $K \subset I^{M+1}$. Then, $\{z\}$ can be interpreted as a homology class in $H_k(I^{M+1}, \mathscr{Z}_2) = H_k(\mathscr{W}^\infty \cup \mathscr{V}_\epsilon, \mathscr{Z}_2)$. Let

$$\mathscr{L} = I^{\epsilon_1} \cup \mathscr{D}^\infty_{M+1}. \tag{2.5}$$

Note that $\mathscr{D}_{M+1} = \mathscr{D}_a$. $\mathscr{W}^\infty$ has the homology of $\mathscr{L}$. Therefore,

$$H_n(\mathscr{W}^\infty) \equiv H_n(\mathscr{L}). \tag{2.6}$$

Since $\mathscr{V}_\epsilon$, $\mathscr{V}_\epsilon \cap \mathscr{W}^\infty$ and $\mathscr{W}^\infty$ are closed A.N.R.'s, $(\mathscr{V}_\epsilon, \mathscr{W}^\infty)$ is an excisive couple and the Mayer-Vietoris sequence holds:

(2.7)
$$\begin{aligned} \dots \longrightarrow H_n(\mathscr{W}^\infty \cap \mathscr{V}_\epsilon) \longrightarrow H_n(\mathscr{W}^\infty) \oplus H_n(\mathscr{V}_\epsilon) \longrightarrow H_n(\mathscr{W}^\infty \cup \mathscr{V}_\epsilon) \\ \longrightarrow H_{n-1}(\mathscr{W}^\infty \cap \mathscr{V}_\epsilon) \longrightarrow \dots . \end{aligned}$$

If $n > m$, $H_n(\mathscr{V}_\epsilon) = H_n(\mathscr{W}^\infty \cap \mathscr{V}_\epsilon) = 0$. Hence, for $n \geq m+1$,

$$H_n(\mathscr{W}^\infty \cup \mathscr{V}_\epsilon) = H_n(\mathscr{W}^\infty) = H_n(\mathscr{L}). \tag{2.8}$$

Since $k \geq m+1$, $\{z\} \in H_k(\mathscr{L})$.

Now, we recall the construction provided in [19] of the new systems of coordinates near each critical point at infinity.

M being chosen as above, we choose $c > 0$ such that: in a neighborhood of the classical critical points for the 3-body problem corresponding to the indices (i, j, k), under the level $M+1$, we have

$$\|[q_i - q_j]\| \leq \frac{1}{4}c; \quad \|[q_i - q_k]\| \leq \frac{1}{4}c; \quad \|[q_j - q_k]\| \leq \frac{1}{4}c,$$

all points on the unstable manifolds of the critical points for the 2-body problem corresponding to the indices (i, j), between the levels $\frac{1}{2}\epsilon_1$ and $M+1$, satisfy

$$\|[q_i - q_j]\| \leq \frac{1}{4}c,$$

and all points on the unstable manifolds of the critical points for the 3-body problem corresponding to the indices (i, j, k), between the levels $\frac{1}{2}\epsilon_1$ and

$M+1$, satisfy

$$\|[q_i - q_j]\| \le \frac{1}{4}c \ \text{ or } \ \|[q_i - q_k]\| \le \frac{1}{4}c \ \text{ or } \ \|[q_j - q_k]\| \le \frac{1}{4}c.$$

Then, by [19] (Proposition 3.1), there exists an $\alpha(M+1) > 0$ such that if $(q_1, q_2) \in \Lambda_{12}$ satisfies

(i) $q_1, q_2 \in I_{12}^{M+1}$,
(ii) $\frac{1}{1+\|[q_1-q_2]\|^2} \le \alpha(M+1)$,

then we have a new system of coordinates (P_1, P_2) such that

$$I_{12}(q_1, q_2) = \frac{1}{2}\int_0^1 |\dot{P}_1|^2 dt + \frac{1}{2}\int_0^1 |\dot{P}_2|^2 dt + \frac{1}{1+\|[P_1 - P_2]\|^2},$$

by [19] (Proposition 3.2), there exists an $\alpha(c, M+1) > 0$ such that if $(q_1, q_2, q_3) \in \Lambda_{123}$ satisfies

(i) $q_1, q_2, q_3 \in I_{123}^{M+1}$,
(ii) $\|[q_1 - q_2]\| \le c$,
(iii) $\frac{1}{1+\|[q_3 - \frac{q_1+q_2}{2}]\|^2} \le \alpha(c, M+1)$,

then we have a new system of coordinates (q_1, q_2, Q_3) such that

$$I_{123}(q_1, q_2, q_3) = I_{12}(q_1, q_2) + \frac{1}{2}\int_0^1 |\dot{Q}_3|^2 dt + \frac{1}{1+\|[Q_3 - \frac{q_1+q_2}{2}]\|^2},$$

by [19] (Proposition 3.5), there exists a $\delta(c, M+1) > 0$ such that if $q \in \Lambda$ satisfies

(i) $q \in I^{M+1}$,
(ii) $\|[q_1 - q_2]\| \le c$,
(iii) $\frac{1}{1+\|[q_3-\frac{q_1+q_2}{2}]\|^2} + \frac{1}{1+\|[q_4-\frac{q_1+q_2}{2}]\|^2} + \frac{1}{1+\|[q_4-q_3]\|^2} \le \delta(c, M+1)$,

then we have a new system of coordinates (q_1, q_2, R_3, R_4) such that

$$\begin{aligned} I(q) = I_{12}(q_1, q_2) &+ \frac{1}{2}\int_0^1 |\dot{R}_3|^2\, dt + \frac{1}{2}\int_0^1 |\dot{R}_4|^2\, dt \\ &+ \frac{1}{1+\|[R_3 - \frac{q_1+q_2}{2}]\|^2} + \frac{1}{1+\|[R_4 - \frac{q_1+q_2}{2}]\|^2} + \frac{1}{1+\|[R_4 - R_3]\|^2}. \end{aligned}$$

We choose $\delta(c, M+1)$ such that

$$\delta(c, M+1) \le -\int_0^1 (\widetilde{V}_{13} + \widetilde{V}_{23})$$

when $\frac{1}{\|[q_3-\frac{q_1+q_2}{2}]\|^2+1} \ge \alpha(c, M+1)$ and

$$\delta(c, M+1) \le -\int_0^1 \widetilde{V}_{34}$$

when $\frac{1}{\|[q_3-q_4]\|^2+1} \ge \alpha(M+1)$.

Then δ_1 is chosen such that

$$0 < \delta_1 < \frac{1}{4}\delta(c, M+1).$$

We then choose c_1 such that $c_1 \geq \sqrt{\frac{1}{\alpha(c, M+1)} - 1}$, $c_1 \geq \sqrt{\frac{1}{\alpha(M+1)} - 1}$ and $\|[q_3 - \frac{q_1+q_2}{2}]\| \leq c_1$ when the coordinates (q_1, q_2, Q_3) are not available or when $\frac{1}{1+\|[Q_3-\frac{q_1+q_2}{2}]\|^2} \geq \frac{1}{2}\delta_1$ and $\|[q_3 - q_4]\| \leq c_1$ when the coordinates (P_3, P_4) are not available or when $\frac{1}{1+\|[P_3-P_4]\|^2} \geq \frac{1}{2}\delta_1$.

Then, by [19] (Proposition 3.3), there exists a $\beta(c, c_1, M+1) > 0$ such that if $q = (q_1, q_2, q_3, q_4) \in \Lambda$ satisfies

(i) $q \in I^{M+1}$,
(ii) $\|[q_1 - q_2]\| \leq c$,
(iii) $\|[q_3 - \frac{q_1+q_2}{2}]\| \leq c_1$,
(iv) $\frac{1}{1+\|[q_4-\frac{q_1+q_2+q_3}{3}]\|^2} \leq \beta(c, c_1, M+1)$,

then we have a new system of coordinates (q_1, q_2, q_3, S_4) such that

$$I(q) = I_{123}(q_1, q_2, q_3) + \frac{1}{2}\int_0^1 |\dot{S}_4|^2 dt + \frac{1}{1 + \|[S_4 - \frac{q_1+q_2+q_3}{3}]\|^2},$$

and by [19] (Proposition 3.4), there exists a $\gamma(c, c_1, M+1) > 0$ such that if $q = (q_1, q_2, q_3, q_4) \in \Lambda$ satisfies

(i) $q \in I^{M+1}$,
(ii) $\|[q_1 - q_2]\| \leq c$,
(iii) $\|[q_3 - q_4]\| \leq c_1$,
(iv) $\frac{1}{1+\|[\frac{q_3+q_4}{2} - \frac{q_1+q_2}{2}]\|^2} \leq \gamma(c, c_1, M+1)$,

then we have a new system of coordinates (q_1, q_2, T_3, T_4) such that

$$I(q) = I_{12}(q_1, q_2) + I_{34}(T_3, T_4) + \frac{1}{1 + |\frac{[T_3+T_4]}{2} - \frac{[q_1+q_2]}{2}|^2}.$$

In the sequel, we denote by $\beta = \beta(c, c_1, M+1)$, $\gamma = \gamma(c, c_1, M+1)$ and $\delta = \delta(c, M+1)$. We choose β and γ such that

$$\beta < \frac{1}{2}\delta_1 \quad \text{and} \quad \gamma < \frac{1}{2}\delta_1.$$

We then choose β_1 and γ_1 such that

$$\beta_1 < \frac{1}{4}\beta \quad \text{and} \quad \gamma_1 < \frac{1}{4}\gamma.$$

We choose all the constants in such a way that they stay valid under any permutation of the indices and such that the transformations between the initial coordinates and the new ones are diffeomorphisms (see [19] for details).

In the sequel, we denote by $W_u^\infty(\bar{q}_i, \bar{q}_j, \bar{q}_k)$ the unstable manifold of the critical set at infinity defined by

$$\{(\bar{q}_i, \bar{q}_j, \bar{q}_k)\} \times \{S_r \in \Re^\ell \,/\, \frac{1}{1 + |S_r - \frac{[\bar{q}_i+\bar{q}_j+\bar{q}_k]}{3}|^2} \leq \beta_1\}$$

where $(\bar{q}_i, \bar{q}_j, \bar{q}_k) \in K_{ijk}^{M+1}$, the set of classical critical points under the level $M+1$ for the 3-body problem corresponding to the indices (i, j, k) and $\{r\} = \{1, 2, 3, 4\} - \{i, j, k\}$, $W_u^\infty((\bar{q}_i, \bar{q}_j); (\bar{T}_k, \bar{T}_r))$ the unstable manifold of the critical set at infinity defined by

$$\{(\bar{q}_i, \bar{q}_j)\} \times \{(\bar{T}_k, \bar{T}_r)\} \times \{[\frac{\bar{T}_k + \bar{T}_r}{2} - \frac{\bar{q}_i + \bar{q}_j}{2}] \in \Re^\ell \,/\, \frac{1}{1 + \|[\frac{\bar{T}_k+\bar{T}_r}{2} - \frac{\bar{q}_i+\bar{q}_j}{2}]\|^2} \leq \gamma_1\}$$

where $(\bar{q}_i, \bar{q}_j) \in K_{ij}^{M+1}$ and $(\bar{T}_k, \bar{T}_r) \in K_{kr}^{M+1}$, the sets of critical points under the level $M+1$ for the 2-body problems corresponding to the indices (i, j) and (k, r), respectively, and such that $\widetilde{I}_{ij}(\bar{q}_i, \bar{q}_j) + \widetilde{I}_{kr}(\bar{T}_k, \bar{T}_r) \leq M+1$, and $\{k, r\} = \{1, 2, 3, 4\} - \{i, j\}$, $W_u^\infty(\bar{q}_i, \bar{q}_j)$ the unstable manifold of the critical set at infinity defined by

$$\{(\bar{q}_i, \bar{q}_j)\} \times \{(R_k, R_s) \in (\Re^\ell)^2 \,/\, \frac{1}{1 + |R_k - \frac{[\bar{q}_i+\bar{q}_j]}{2}|^2} + \frac{1}{1 + |R_s - \frac{[\bar{q}_i+\bar{q}_j]}{2}|^2} + \frac{1}{1 + |R_k - R_s|^2} \leq \delta_1\}$$

where $(\bar{q}_i, \bar{q}_j) \in K_{ij}^{M+1}$, the set of critical points under the level $M+1$ for the 2-body problem corresponding to the indices (i, j) and $\{k, s\} = \{1, 2, 3, 4\} - \{i, j\}$.

Note that the translational symmetry of I, and in particular of I_{ij}, allows us to define the variable $[\frac{T_k+T_r}{2} - \frac{q_i+q_j}{2}] \in \Re^\ell$ independently of $(q_i, q_j) \in \widetilde{\Lambda}_{ij}$ and $(T_k, T_r) \in \widetilde{\Lambda}_{kr}$.

Then, we have the following lemma.

Lemma 2.2. *c, c_1 being chosen as above, on $W_u^\infty(\bar{q}_i, \bar{q}_j, \bar{q}_k) - \operatorname{int} I^{\epsilon_1}$, we have*
$\|[q_i - q_j]\| \leq \frac{1}{4}c$ and $\|[q_k - \frac{q_i+q_j}{2}]\| \leq c_1$ or
$\|[q_i - q_k]\| \leq \frac{1}{4}c$ and $\|[q_j - \frac{q_i+q_k}{2}]\| \leq c_1$ or
$\|[q_j - q_k]\| \leq \frac{1}{4}c$ and $\|[q_i - \frac{q_j+q_k}{2}]\| \leq c_1$, and
on $W_u^\infty((\bar{q}_i, \bar{q}_j); (\bar{T}_k, \bar{T}_r)) - \operatorname{int} I^{\epsilon_1}$, we have
$\|[q_i - q_j]\| \leq \frac{1}{4}c$ and $\|[T_k - T_r]\| \leq c_1$ or
$\|[T_k - T_r]\| \leq \frac{1}{4}c$ and $\|[q_i - q_j]\| \leq c_1$.

Proof. By the choice of c, all points on the unstable manifolds along $\widetilde{Z}_{ijk}$ (pseudogradient vector field constructed in [19], relying on [6], for the 3-body problem corresponding to the indices (i, j, k)) of the classical critical points of $\widetilde{I}_{ijk}$ between the levels ϵ_1 and $M+1$ satisfy

$$\|[q_i - q_j]\| \leq \frac{1}{4}c \text{ or } \|[q_i - q_k]\| \leq \frac{1}{4}c \text{ or } \|[q_j - q_k]\| \leq \frac{1}{4}c.$$

This is possible since on $W_u(\bar{q}_i, \bar{q}_j, \bar{q}_k) - \operatorname{int} \widetilde{I}_{ijk}^{\epsilon_1}$ only one body can split away from the others.

On $W_u^\infty(\bar{q}_i, \bar{q}_j, \bar{q}_k)$, by the choice of c, every trajectory starts in a neighborhood where

$$\|[q_i - q_j]\| \leq \frac{1}{4}c\,, \quad \|[q_i - q_k]\| \leq \frac{1}{4}c \text{ and } \|[q_j - q_k]\| \leq \frac{1}{4}c\,,$$

then, if q_k goes away from (q_i, q_j) and enters the region where the coordinates (q_i, q_j, Q_k) are available, the flow of $\widetilde{Z}_{ijk}$ stays in the region where

$$\frac{1}{1+\|[Q_k-\frac{q_i+q_j}{2}]\|^2}+\frac{1}{1+\|[S_r-\frac{q_i+q_j+q_k}{3}]\|^2}\geq\delta_1$$

since otherwise we are in the region where the coordinates (q_i, q_j, R_k, R_r) are available and

$$\frac{1}{1+\|[R_k-\frac{q_i+q_j}{2}]\|^2}+\frac{1}{1+\|[R_r-\frac{q_i+q_j}{2}]\|^2}+\frac{1}{1+\|[R_k-R_r]\|^2}\leq\delta_1$$

which defines another critical set at infinity. We also have

$$\frac{1}{1+\|[S_r-\frac{q_i+q_j+q_k}{3}]\|^2}\leq\beta_1$$

on $W_u^\infty(\bar{q}_i, \bar{q}_j, \bar{q}_k)$. But then,

$$\frac{1}{1+\|[Q_k-\frac{q_i+q_j}{2}]\|^2}\geq\delta_1-\beta_1>\frac{1}{2}\delta_1$$

and by the definition of c_1,

$$\|[q_k-\frac{q_i+q_j}{2}]\|\leq c_1.$$

Next, by the choice of c, all points on the unstable manifolds along $\widetilde{Z}_{ij}$ (pseudogradient vector field constructed in [19] for the 2-body problem corresponding to the indices (i, j)) of the critical points of $\widetilde{I}_{ij}$ between the levels ϵ_1 and $M+1$ satisfy

$$\|[q_i-q_j]\|\leq\frac{1}{4}c.$$

Then, if q_k and q_r split away from each other and enter the region where the coordinates (P_k, P_r) are available, the flow of $\widetilde{Z}_{kr}$ stays in the region where

$$\frac{1}{1+\|[P_k-P_r]\|^2}+\frac{1}{1+\|[\frac{P_k+P_r}{2}-\frac{q_i+q_j}{2}]\|^2}\geq\delta_1$$

since otherwise we are in the region where the coordinates (q_i, q_j, R_k, R_r) are available and

$$\frac{1}{1+\|[R_k-\frac{q_i+q_j}{2}]\|^2}+\frac{1}{1+\|[R_r-\frac{q_i+q_j}{2}]\|^2}+\frac{1}{1+\|[R_k-R_r]\|^2}\leq\delta_1.$$

We also have

$$\frac{1}{1+\|[\frac{P_k+P_r}{2}-\frac{q_i+q_j}{2}]\|^2}\leq\gamma_1$$

on $W_u^\infty((\bar{q}_i, \bar{q}_j); (\bar{T}_k, \bar{T}_r))$. But then,

$$\frac{1}{1+\|[P_k-P_r]\|^2}\geq\delta_1-\gamma_1>\frac{1}{2}\delta_1$$

and by the definition of c_1,

$$\|[q_k-q_r]\|\leq c_1. \quad \square$$

Lemma 2.3. (1) *There exists a* C^∞ *function* $\beta'(q_1, q_2, q_3)$ *such that if* $q_4 \in \Re^\ell$ *satisfies* (iv) *of Proposition* 3.3 *in* [19] *with* $\beta'(q_1, q_2, q_3)$ *then both systems of coordinates given by Proposition* 3.3 *in* [19] *are available at* (q_1, q_2, q_3, q_4).

(2) *There exists a* C^∞ *function* $\gamma'((q_1, q_2); (q_3, q_4))$ *such that if* $[\frac{q_3+q_4}{2} - \frac{q_1+q_2}{2}]$ *satisfies* (iv) *of Proposition* 3.4 *in* [19] *with* $\gamma'((q_1, q_2); (q_3, q_4))$ *then both systems of coordinates given by Proposition* 3.4 *in* [19] *are available at* (q_1, q_2, q_3, q_4).

(3) *There exists a* C^∞ *function* $\delta'(q_1, q_2)$ *such that if* $(q_3, q_4) \in (\Re^\ell)^2$ *satisfies* (iii) *of Proposition* 3.5 *in* [19] *with* $\delta'(q_1, q_2)$ *then both systems of coordinates given by Proposition* 3.5 *in* [19] *are available at* (q_1, q_2, q_3, q_4).

Moreover, we can choose $\beta'(q_1, q_2, q_3)$, $\gamma'((q_1, q_2); (q_3, q_4))$ *and* $\delta'(q_1, q_2)$ *such that*

$$\beta'(q_1, q_2, q_3) = \beta_1 \text{ on } \bigcup_{(q_1, q_2, q_3) \in K_{123}^{M+1}} W_u(\bar{q}_1, \bar{q}_2, \bar{q}_3) - \operatorname{int} I_{123}^{\epsilon_1},$$

$$\gamma'((q_1, q_2); (q_3, q_4)) = \gamma_1 \text{ on}$$

$$\bigcup_{\substack{(q_1, q_2) \in K_{12} \\ (q_3, q_4) \in K_{34} \\ \tilde{I}_{12}(q_1, q_2) + \tilde{I}_{34}(q_3, q_4) \le M+1}} (W_u(\bar{q}_1, \bar{q}_2) \times W_u(\bar{q}_3, \bar{q}_4)) - \operatorname{int}(I_{12} + I_{34})^{\epsilon_1}, \tag{2.9}$$

$$\delta'(q_1, q_2) = \delta_1 \text{ on } \bigcup_{(q_1, q_2) \in K_{12}^{M+1}} W_u(\bar{q}_1, \bar{q}_2) - \operatorname{int} I_{12}^{\epsilon_1}$$

and

(2.10)

$$\beta'(q_1, q_2, q_3) < \inf\left(-\int_0^1 (\tilde{V}_{12} + \tilde{V}_{13}); -\int_0^1 (\tilde{V}_{12} + \tilde{V}_{23}); -\int_0^1 (\tilde{V}_{13} + \tilde{V}_{23})\right)$$

$$\gamma'((q_1, q_2); (q_3, q_4)) < \inf\left(-\int_0^1 \tilde{V}_{12}; -\int_0^1 \tilde{V}_{34}\right).$$

Proof. The proof is similar to the one given by Bahri-Rabinowitz [6] for the 3-body problem. We recall it here. By [19] (Proposition 3.3), for any $(q_1, q_2, q_3) \in \Lambda_{123}$ and $c(q_1, q_2) = \|[q_1 - q_2]\|$, $c_1(q_1, q_2, q_3) = \|[q_3 - \frac{q_1+q_2}{2}]\|$, $M = I_{123}(q_1, q_2, q_3) + 1$, there is a $\beta(2c, 2c_1, 2M)$ for which the conclusions of Proposition 3.3 in [19] are valid at (q_1, q_2, q_3, q_4) if $q_4 \in \Re^\ell$ satisfies (iv).

We take

$$\beta'(2c, 2c_1, 2M) < \frac{1}{4}\beta(2c, 2c_1, 2M)$$

such that

$$\beta'(2c, 2c_1, 2M) < -\int_0^1 (\tilde{V}_{13} + \tilde{V}_{23})$$

on the set defined by $\|[q_1' - q_2']\| \le 2c$ and $\|[q_3' - \frac{q_1'+q_2'}{2}]\| \le 2c_1$.

We can permute the indices $(1, 2, 3)$ and take the smallest of the 3 constants provided this way.

In a neighborhood $W_{(q_1', q_2', q_3')}$ of (q_1', q_2', q_3'), we have $I_{123}(q_1, q_2, q_3) \le 2I_{123}(q_1', q_2', q_3')$, $\|[q_i - q_j]\| \le 2c(q_i', q_j')$ and $\|[q_k - \frac{q_i+q_j}{2}]\| \le 2c_1(q_i', q_j', q_k')$ for any permutation (i, j, k) of the indices $(1, 2, 3)$. This gives us a covering of Λ_{123} which possesses a locally finite refinement W_m. Let (ρ_m) be a smooth partition of unity subordinate to W_m. Define

$$\beta'(q_1, q_2, q_3) = \sum_m \beta'_m \rho_m(q_1, q_2, q_3)$$

where β'_m is the $\beta'(2c, 2c_1, 2M)$ associated to some $W_{(q'_1, q'_2, q'_3)}$ such that $W_m \subset W_{(q'_1, q'_2, q'_3)}$.

Then

$$\beta'(q_1, q_2, q_3) \leq \beta'_W$$

where β'_W is the largest of the β'_m such that $\rho_m(q_1, q_2, q_3) \neq 0$. Since Proposition 3.3 in [19] holds at $(q_1, q_2, q_3, q_4) \in \Lambda$ where $q_4 \in \Re^\ell$ satisfies (iv) with β'_W, it holds a fortiori for a subclass of $q_4 \in \Re^\ell$ satisfying (iv) with $\beta'(q_1, q_2, q_3)$ and $\beta'(q_1, q_2, q_3)$ satisfies (2.10) since β'_W does so.

In the same way, we define $\gamma'((q_1, q_2); (q_3, q_4))$ and $\delta'(q_1, q_2)$.

Now, Lemma 2.2 implies that we can take one of the neighborhoods W_m large enough so that

$$\Big(\bigcup_{(\bar q_1, \bar q_2, \bar q_3) \in K_{123}^{M+1}} W_u(\bar q_1, \bar q_2, \bar q_3) - \operatorname{int} I_{123}^{\epsilon_1}\Big) \subset W_m$$

and

$$\Big(\bigcup_{(\bar q_1, \bar q_2, \bar q_3) \in K_{123}^{M+1}} W_u(\bar q_1, \bar q_2, \bar q_3) - \operatorname{int} I_{123}^{\epsilon_1}\Big) \cap W_{m'} = \varnothing$$

for any $m' \neq m$. Then we take $\beta'_m = \beta_1$ for this neighborhood.

Doing the same thing for the other unstable manifolds at infinity will imply (2.9).

Let

$$\mathscr{M}_{123}^{\epsilon_1} = \Big\{(q_1, q_2, q_3, S_4) \in \Lambda_{123} \times \Re^\ell / I_{123}(q_1, q_2, q_3) + \frac{1}{1+|S_4 - [\frac{q_1+q_2+q_3}{3}]|^2} \leq \epsilon_1 ;\ \frac{1}{1+|S_4 - [\frac{q_1+q_2+q_3}{3}]|^2} \leq \beta'(q_1, q_2, q_3)\Big\}.$$

$\mathscr{M}_{123}^{\epsilon_1}$ is a trivializable sphere bundle over $I_{123}^{\epsilon_1}$ with fiber

$$F(q_1, q_2, q_3) = \Big\{S_4 \in \Re^\ell / \frac{1}{1+|S_4 - [\frac{q_1+q_2+q_3}{3}]|^2} \leq \inf(\epsilon_1 - I_{123}(q_1, q_2, q_3), \beta'(q_1, q_2, q_3))\Big\}.$$

Since, by [6], $I_{123}^{\epsilon_1}$ has the homotopy type of a submanifold of $(\Re^\ell)^3$, we have

$$H_n(\mathscr{M}_{123}^{\epsilon_1}; \mathscr{Z}_2) = 0 \tag{2.11}$$

for $n \geq 4\ell$.

Now, let

$$\mathscr{M}_{12;34}^{\epsilon_1} = \Big\{(q_1, q_2, T_3, T_4) \in \Lambda_{12} \times \Lambda_{34} / I_{12}(q_1, q_2) + I_{34}(T_3, T_4) + \frac{1}{1+|[\frac{T_3+T_4}{2} - \frac{q_1+q_2}{2}]|^2} \leq \epsilon_1 ;\ \frac{1}{1+|[\frac{T_3+T_4}{2} - \frac{q_1+q_2}{2}]|^2} \leq \gamma'((q_1, q_2); (T_3, T_4))\Big\}.$$

Let $\mathscr{P}_{12;34}$ be the fiber bundle over $I_{12}^{\epsilon_1}$ with fiber

$$F(q_1, q_2) = \{(T_3, T_4) \in \widetilde{\Lambda}_{34} / I_{34}(T_3, T_4) \leq \epsilon_1 - I_{12}(q_1, q_2)\}.$$

$\mathscr{M}^{\epsilon_1}_{12;34}$ is a trivializable sphere bundle over $\mathscr{P}_{12;34}$ with fiber

$$F(q_1, q_2, T_3, T_4) = \{Q = [\tfrac{T_3+T_4}{2} - \tfrac{q_1+q_2}{2}] \in \Re^\ell / \tfrac{1}{1+|Q|^2} \le \inf(\epsilon_1 - I_{12}(q_1, q_2) - I_{34}(T_3, T_4), \gamma'((q_1, q_2); (T_3, T_4)))\}.$$

Since $I^{\epsilon_1}_{12}$ and $F(q_1, q_2)$ have the homotopy type of submanifolds of $(\Re^\ell)^2$, we have

$$H_n(\mathscr{M}^{\epsilon_1}_{12;34}; \mathscr{Z}_2) = 0 \tag{2.12}$$

for $n \ge 5\ell$.

Finally, let

$$\mathscr{M}^{\epsilon_1}_{12} = \{(q_1, q_2, R_3, R_4) \in \Lambda_{12} \times (\Re^\ell)^2 / I_{12}(q_1, q_2) + \tfrac{1}{1+|R_3-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_4-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_3-R_4|^2} \le \epsilon_1 ;\ \tfrac{1}{1+|R_3-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_4-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_3-R_4|^2} \le \delta'(q_1, q_2)\}.$$

$\mathscr{M}^{\epsilon_1}_{12}$ is a trivializable bundle over $I^{\epsilon_1}_{12}$ with fiber

$$G(q_1, q_2) = \{(R_3, R_4) \in (\Re^\ell)^2 / \tfrac{1}{1+|R_3-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_4-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_3-R_4|^2} \le \inf(\epsilon_1 - I_{12}(q_1, q_2), \delta'(q_1, q_2))\}.$$

$G(q_1, q_2)$ fibers over
$G'(q_1, q_2) = \{R_3 \in \Re^\ell / \frac{1}{1+|R_3-[\frac{q_1+q_2}{2}]|^2} \le \inf(\epsilon_1 - I_{12}(q_1, q_2), \delta'(q_1, q_2))\}$
with fiber

$$G(q_1, q_2, R_3) = \{R_4 \in \Re^\ell / \tfrac{1}{1+|R_4-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_3-R_4|^2} \le \inf(\epsilon_1 - I_{12}(q_1, q_2), \delta'(q_1, q_2)) - \tfrac{1}{1+|R_3-[\frac{q_1+q_2}{2}]|^2}\}.$$

$G(q_1, q_2, R_3)$ has the homotopy type of a sphere or the wedge of two spheres in $\Re^\ell$, depending on (q_1, q_2, R_3). Hence,

$$H_n(\mathscr{M}^{\epsilon_1}_{12}; \mathscr{Z}_2) = 0 \tag{2.13}$$

for $n \ge 4\ell - 1$.

Next, let

$$\mathscr{Q}_{123} = \{(q_1, q_2, q_3, S_4) \in \Lambda_{123} \times \Re^\ell / \tfrac{1}{1+|S_4-[\frac{q_1+q_2+q_3}{3}]|^2} \le \beta'(q_1, q_2, q_3)\},$$

$$\mathscr{Q}_{12;34} = \{(q_1, q_2, T_3, T_4) \in \Lambda_{12} \times \Lambda_{34} / \tfrac{1}{1+|[\frac{T_3+T_4}{2} - \frac{q_1+q_2}{2}]|^2} \le \gamma'((q_1, q_2); (T_3, T_4))\},$$

$$\mathscr{Q}_{12} = \{(q_1, q_2, R_3, R_4) \in \Lambda_{12} \times (\Re^\ell)^2 / \tfrac{1}{1+|R_3-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_4-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|R_3-R_4|^2} \le \delta'(q_1, q_2)\}$$

and

$$\mathscr{Q} = I^{\epsilon_1} \cup (\bigcup_{i<j<r} \mathscr{Q}_{ijr}) \cup (\bigcup_{i<j} \mathscr{Q}_{ij;rs}) \cup (\bigcup_{i<j} \mathscr{Q}_{ij}). \tag{2.14}$$

Lemma 2.4.

$$\mathscr{L} \subset \mathscr{Q}$$

and $\{z\}$ *can be represented by a chain having support in* $\mathscr{Q}$.

Proof. This is a result of Lemma 2.2, Lemma 2.3 and the structure of the unstable manifolds at infinity given by [19] (Theorem 5.1).

Therefore, $\{z\}$ can be considered to be a homology class in $H_k(\mathscr{C}; \mathscr{Z}_2)$. We have

$$\begin{aligned} \mathscr{C}_{ijr} \cap I^{\epsilon_1} &= \mathscr{M}_{ijr}^{\epsilon_1}, \\ \mathscr{C}_{ij;rs} \cap I^{\epsilon_1} &= \mathscr{M}_{ij;rs}^{\epsilon_1}, \\ \mathscr{C}_{ij} \cap I^{\epsilon_1} &= \mathscr{M}_{ij}^{\epsilon_1}. \end{aligned} \tag{2.15}$$

Lemma 2.5. *The only non-empty intersections between the sets* $\mathscr{C}_{ijr}$, $\mathscr{C}_{ij;rs}$, $\mathscr{C}_{ij}$ *are*

$$\begin{aligned} \mathscr{R}_{12;3} &= \mathscr{C}_{123} \cap \mathscr{C}_{12} \\ &= \{(q_1, q_2, Q_3, S_4) \in \Lambda_{12} \times (\Re^\ell)^2 / \tfrac{1}{1+|Q_3-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|S_4-[\frac{q_1+q_2+q_3}{3}]|^2} \\ &\qquad \le \delta'(q_1, q_2);\ \tfrac{1}{1+|S_4-[\frac{q_1+q_2+q_3}{3}]|^2} \le \beta'(q_1, q_2, q_3)\}, \\ \mathscr{R}_{12;34} &= \mathscr{C}_{12;34} \cap \mathscr{C}_{12} \\ &= \{(q_1, q_2, P_3, P_4) \in \Lambda_{12} \times (\Re^\ell)^2 / \tfrac{1}{1+|\frac{P_3+P_4}{2}-[\frac{q_1+q_2}{2}]|^2} + \tfrac{1}{1+|P_4-P_3|^2} \\ &\qquad \le \delta'(q_1, q_2);\ \tfrac{1}{1+|\frac{P_3+P_4}{2}-[\frac{q_1+q_2}{2}]|^2} \le \gamma'((q_1, q_2); (P_3, P_4))\} \end{aligned}$$

and the ones obtained by permutation of the indices.

Proof. It is enough to show that $\mathscr{C}_{123} \cap \mathscr{C}_{12;34} = \varnothing$.

On $\mathscr{C}_{123}$, we have

$$\frac{1}{1+\|[S_4 - \frac{q_1+q_2+q_3}{3}]\|^2} \le \beta'(q_1, q_2, q_3).$$

But

$$\frac{1}{1+\|[S_4 - \frac{q_1+q_2+q_3}{3}]\|^2} = -\int_0^1 (\widetilde{V}_{14} + \widetilde{V}_{24} + \widetilde{V}_{34})$$

and by (2.10)

$$\beta'(q_1, q_2, q_3) < -\int_0^1 (\widetilde{V}_{13} + \widetilde{V}_{23}).$$

Hence, on $\mathscr{C}_{123}$, we have

$$-\int_0^1 (\widetilde{V}_{14} + \widetilde{V}_{24} + \widetilde{V}_{34}) < -\int_0^1 (\widetilde{V}_{13} + \widetilde{V}_{23}).$$

On the other hand, on $\mathscr{C}_{12;34}$, we have

$$\frac{1}{1+\|[\frac{T_3+T_4}{2} - \frac{q_1+q_2}{2}]\|^2} \le \gamma'((q_1, q_2); (T_3, T_4)).$$

But

$$\frac{1}{1+\|[\frac{T_3+T_4}{2} - \frac{q_1+q_2}{2}]\|^2} = -\int_0^1 (\widetilde{V}_{13} + \widetilde{V}_{14} + \widetilde{V}_{23} + \widetilde{V}_{24})$$

and by (2.10)

$$\gamma'((q_1, q_2); (T_3, T_4)) < -\int_0^1 \widetilde{V}_{34}.$$

Hence, on $\mathscr{C}_{12;34}$, we have

$$-\int_0^1 (\widetilde{V}_{13} + \widetilde{V}_{14} + \widetilde{V}_{23} + \widetilde{V}_{24}) < -\int_0^1 \widetilde{V}_{34}.$$

On $\mathscr{Q}_{123} \cap \mathscr{Q}_{12;34}$, we must then have

$$-2 \int_0^1 (\widetilde{V}_{14} + \widetilde{V}_{24}) < 0$$

which is impossible by (V_2).

$\mathscr{R}_{12;3}$ fibers over Λ_{12} with fiber

$$\mathscr{N}(q_1, q_2) = \{(Q_3, S_4) \in (\Re^\ell)^2 / \frac{1}{1+|Q_3-[\frac{q_1+q_2}{2}]|^2} \leq \delta'(q_1, q_2); \frac{1}{1+|S_4-[\frac{q_1+q_2+q_3}{3}]|^2} \leq \inf(\beta'(q_1, q_2, q_3), \delta'(q_1, q_2) - \frac{1}{1+|Q_3-[\frac{q_1+q_2}{2}]|^2})\}.$$

$\mathscr{N}(q_1, q_2)$ fibers over the exterior of a ball with fiber also equal to the exterior of a ball.

$\mathscr{R}_{12;34}$ fibers over Λ_{12} with fiber

$$\mathscr{N}'(q_1, q_2) = \{(P_3, P_4) \in (\Re^\ell)^2 / \frac{1}{1+|P_3-P_4|^2} \leq \delta'(q_1, q_2); \frac{1}{1+|\frac{P_3+P_4}{2}-[\frac{q_1+q_2}{2}]|^2} \leq \inf(\gamma'((q_1, q_2); (P_3, P_4)), \delta'(q_1, q_2) - \frac{1}{1+|P_3-P_4|^2})\}.$$

$\mathscr{N}'(q_1, q_2)$ fibers over the exterior of a ball with fiber also equal to the exterior of a ball.

Let

$$\mathscr{A} = \bigcup_{i<j<r} (\mathscr{Q}_{ijr} \cup I^{\epsilon_1}),$$

$$\mathscr{B} = \bigcup_{i<j} (\mathscr{Q}_{ij;rs} \cup I^{\epsilon_1}),$$

$$\mathscr{C} = \bigcup_{i<j} (\mathscr{Q}_{ij} \cup I^{\epsilon_1}),$$

then

$$\mathscr{Q} = \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}.$$

Lemma 2.5 implies that

$$\mathscr{A} \cap \mathscr{B} = I^{\epsilon_1},$$

$$\mathscr{A} \cap \mathscr{C} = \bigcup_{i,j,r} (\mathscr{R}_{ij;r} \cup I^{\epsilon_1})$$

and

$$\mathscr{B} \cap \mathscr{C} = \bigcup_{i,j} (\mathscr{R}_{ij;rs} \cup I^{\epsilon_1}).$$

We can choose $\delta'(q_i, q_j)$ such that the sets $\mathscr{R}_{ij;r} - I^{\epsilon_1}$ are disjoint and the sets $\mathscr{R}_{ij;rs} - I^{\epsilon_1}$ are disjoint.

Since $H_k(I^{\epsilon_1}) = 0$ for $k \geq 4\ell + 1$, we can show, using Mayer-Vietoris sequences, that for $k \geq 4\ell + 1$,

$$H_k(\mathscr{A} \cap \mathscr{C}) = \bigoplus_{i,j,r} H_k(\mathscr{R}_{ij;r} \cup I^{\epsilon_1}) \tag{2.16}$$

and

$$H_k(\mathscr{B} \cap \mathscr{C}) = \bigoplus_{i,j} H_k(\mathscr{R}_{ij;rs} \cup I^{\epsilon_1}). \tag{2.17}$$

We can also show that for $k \geq 4\ell - 1$ we have

$$H_k(\mathscr{R}_{ij;r} \cap I^{\epsilon_1}) = 0 \tag{2.18}$$

and

$$H_k(\mathscr{R}_{ij;rs} \cap I^{\epsilon_1}) = 0. \tag{2.19}$$

Therefore, again using Mayer-Vietoris sequences, we can show that

$$H_k(\mathscr{A} \cap \mathscr{C}) = \bigoplus_{i,j,r} H_k(R_{ij;r}) \tag{2.20}$$

and

$$H_k(\mathscr{B} \cap \mathscr{C}) = \bigoplus_{i,j} H_k(R_{ij;rs}). \tag{2.21}$$

We can check that the couple $(\mathscr{A}, \mathscr{B} \cup \mathscr{C})$ is excisive and the Mayer-Vietoris sequence holds:

$$\begin{aligned} \dots H_k(\mathscr{A} \cap \mathscr{C}) \longrightarrow H_k(\mathscr{A}) \oplus H_k(\mathscr{B} \cup \mathscr{C}) \longrightarrow H_k(\mathscr{A} \cup \mathscr{B} \cup \mathscr{C}) \\ \longrightarrow H_{k-1}(\mathscr{A} \cap \mathscr{C}) \longrightarrow \dots . \end{aligned}$$

$(\mathscr{B}, \mathscr{C})$ is also an excisive couple. Hence, the following Mayer-Vietoris sequence holds:

$$\dots H_k(\mathscr{B} \cap \mathscr{C}) \longrightarrow H_k(\mathscr{B}) \oplus H_k(\mathscr{C}) \longrightarrow H_k(\mathscr{B} \cup \mathscr{C}) \longrightarrow H_{k-1}(\mathscr{B} \cap \mathscr{C}) \longrightarrow \dots .$$

We conclude that

$$\operatorname{rank} H_k(\mathscr{B} \cup \mathscr{C}) \leq \operatorname{rank} H_k(\mathscr{B}) + \operatorname{rank} H_k(\mathscr{C}) + \operatorname{rank} H_{k-1}(\mathscr{B} \cap \mathscr{C})$$

and

$$\begin{aligned} \operatorname{rank} H_k(\mathscr{A} \cup \mathscr{B} \cup \mathscr{C}) &\leq \operatorname{rank} H_k(\mathscr{A}) + \operatorname{rank} H_k(\mathscr{B} \cup \mathscr{C}) + \operatorname{rank} H_{k-1}(\mathscr{A} \cap \mathscr{C}) \\ &\leq \operatorname{rank} H_k(\mathscr{A}) + \operatorname{rank} H_k(\mathscr{B}) + \operatorname{rank} H_k(\mathscr{C}) \\ &\quad + \operatorname{rank} H_{k-1}(\mathscr{A} \cap \mathscr{C}) + \operatorname{rank} H_{k-1}(\mathscr{B} \cap \mathscr{C}). \end{aligned}$$

We can choose the functions $\beta(q_i, q_j, q_r)$, $\gamma((q_i, q_j); (q_r, q_s))$ and $\delta(q_i, q_j)$ such that the sets $\mathscr{Q}_{ijr} - I^{\epsilon_1}$ are disjoint, the sets $\mathscr{Q}_{ij;rs} - I^{\epsilon_1}$ are disjoint and the sets $\mathscr{Q}_{ij} - I^{\epsilon_1}$ are disjoint. Hence, using Mayer-Vietoris sequences, (2.11), (2.12), (2.13), (2.15) and the fact that $H_k(I^{\epsilon_1}) = 0$ for $k \geq 4\ell + 1$, we show that, for $k \geq \max(5\ell, m+1)$, we have

$$H_k(\mathscr{A}) = \bigoplus_{i<j<r} H_k(\mathscr{Q}_{ijr}),$$

$$H_k(\mathscr{B}) = \bigoplus_{i<j} H_k(\mathscr{Q}_{ij;rs}),$$

$$H_k(\mathscr{C}) = \bigoplus_{i<j} H_k(\mathscr{Q}_{ij}).$$

$\mathscr{Q}_{ijr}$ is a trivializable bundle over Λ_{ijr} with fiber equal to the exterior of a ball.

$\mathscr{Q}_{ij;rs}$ is a trivializable bundle over $\Lambda_{ij} \times \Lambda_{rs}$ with fiber equal to the exterior of a ball.

$\mathscr{Q}_{ij}$ is a trivializable bundle over Λ_{ij} with fiber $S(q_i, q_j)$ given by

$S(q_i, q_j) = \{(R_k, R_s) \in (\Re^\ell)^2 \ / \ \frac{1}{1+|R_k-[\frac{q_i+q_j}{2}]|^2} + \frac{1}{1+|R_s-[\frac{q_i+q_j}{2}]|^2} + \frac{1}{1+|R_k-R_s|^2} \leq \delta'(q_i, q_j)\}$.

$S(q_i, q_j)$ is itself a trivializable bundle over the exterior of a ball with fiber having the homotopy type of a sphere or the wedge of two spheres in $\Re^\ell$.

Therefore,

$$\text{rank}\, H_k(\mathscr{Q}_{ijr}) = \text{rank}\, H_{k-(\ell-1)}(\Lambda_{ijr}) + \text{rank}\, H_k(\Lambda_{ijr}), \tag{2.22}$$

$$\begin{aligned}\text{rank}\, H_k(\mathscr{Q}_{ij;rs}) &= \text{rank}\, H_{k-(\ell-1)}(\Lambda_{ij} \times \Lambda_{rs}) + \text{rank}\, H_k(\Lambda_{ij} \times \Lambda_{rs}) \\ &= \sum_{n=0}^{k-(\ell-1)} \text{rank}\, H_{k-(\ell-1)-n}(\Lambda_{ij}) \times \text{rank}\, H_n(\Lambda_{ij}) \\ &\quad + \sum_{n=0}^{k} \text{rank}\, H_{k-n}(\Lambda_{ij}) \times \text{rank}\, H_n(\Lambda_{ij}),\end{aligned} \tag{2.23}$$

(2.24)
$$\text{rank}\, H_k(\mathscr{Q}_{ij}) \leq 2\,\text{rank}\, H_{k-(2\ell-2)}(\Lambda_{ij}) + 3\,\text{rank}\, H_{k-(\ell-1)}(\Lambda_{ij}) + \text{rank}\, H_k(\Lambda_{ij}),$$

(2.25)
$$\text{rank}\, H_k(\mathscr{R}_{ij;r}) = \text{rank}\, H_{k-(2\ell-2)}(\Lambda_{ij}) + 2\,\text{rank}\, H_{k-(\ell-1)}(\Lambda_{ij}) + \text{rank}\, H_k(\Lambda_{ij}),$$

(2.26)
$$\text{rank}\, H_k(\mathscr{R}_{ij;rs}) = \text{rank}\, H_{k-(2\ell-2)}(\Lambda_{ij}) + 2\,\text{rank}\, H_{k-(\ell-1)}(\Lambda_{ij}) + \text{rank}\, H_k(\Lambda_{ij}).$$

(2.22), (2.23), (2.24), (2.25) and (2.26) are independent of M. We conclude that, for $k \geq \max(5\ell, m+1)$, we have

(2.27)
$$\begin{aligned}\text{rank}\, H_k(\Lambda) \leq\ & 4(\text{rank}\, H_{k-(\ell-1)}(\Lambda_{123}) + \text{rank}\, H_k(\Lambda_{123})) \\ & + 3(\text{rank}\, H_{k-(\ell-1)}(\Lambda_{12} \times \Lambda_{34}) + \text{rank}\, H_k(\Lambda_{12} \times \Lambda_{34})) \\ & + 6(2\,\text{rank}\, H_{k-(2\ell-2)}(\Lambda_{12}) + 3\,\text{rank}\, H_{k-(\ell-1)}(\Lambda_{12}) + \text{rank}\, H_k(\Lambda_{12})) \\ & + 18(\text{rank}\, H_{k-1-(2\ell-2)}(\Lambda_{12}) + 2\,\text{rank}\, H_{k-1-(\ell-1)}(\Lambda_{12}) \\ & + \text{rank}\, H_{k-1}(\Lambda_{12})).\end{aligned}$$

On the other hand, we have the following theorem of E. Fadell and S. Husseini [10]:

Theorem (E. Fadell and S. Husseini). *If $X = \{x_n\}$ denotes a fixed set of mutually distinct points in $\Re^\ell$ and $X_j = \{x_1, \dots, x_j\}$ then, using coefficients in $\mathscr{Z}_2$, we have*

(2.28)
$$H_*(\Omega F_j(\Re^\ell)) = H_*(\Omega(\Re^\ell - X_{j-1})) \otimes H_*(\Omega(\Re^\ell - X_{j-2})) \otimes \dots \otimes H_*(\Omega(\Re^\ell - X_1))$$

where $F_j(\Re^\ell)$ denotes the j-th configuration space on $\Re^\ell$ and ΩY is the space of free loops on Y.

Here, we have $\Lambda = \Omega F_4(\Re^\ell)$, $\Lambda_{123} = \Omega F_3(\Re^\ell)$ and $\Lambda_{12} = \Omega F_2(\Re^\ell)$. By results of Vigué-Poirrier [20] and Fadell-Husseini [10], we have

$$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j)\,;\mathscr{Z}_2) \neq 0 \text{ only if } k \equiv 0\,,\,1 \pmod{\ell - 2},$$

$$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j)\,;\mathscr{Z}_2) \leq j^r \text{ for } k = r(\ell-2) \text{ and } k = r(\ell-2)+1$$

and

$$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j)\,;\mathscr{Z}_2) \geq \frac{j^r}{r(r-1)}(1 - rj^{-\frac{r}{2}})$$
$$\text{for } k = r(\ell-2) \text{ and } k = r(\ell-2)+1.$$

By (2.28), we have

$$H_*(\Lambda_{12}) = H_*(\Omega(\Re^\ell - X_1)),$$

$$H_*(\Lambda_{123}) = H_*(\Omega(\Re^\ell - X_2)) \otimes H_*(\Lambda_{12})$$

and

$$H_*(\Lambda) = H_*(\Omega(\Re^\ell - X_3)) \otimes H_*(\Lambda_{123}).$$

Hence,

$$\operatorname{rank} H_k(\Lambda_{12}) \leq 1$$

and

$$\operatorname{rank} H_k(\Lambda_{12} \times \Lambda_{34}) \leq k+1.$$

If $k = r(\ell-2)+1$, then

$$\operatorname{rank} H_k(\Lambda_{123}) \leq 2^{r+2}$$

and

$$\operatorname{rank} H_{k-(\ell-1)}(\Lambda_{123}) \leq 2^{r+1}.$$

We also have

$$\begin{aligned} \operatorname{rank} H_k(\Lambda) &= \textstyle\sum_{n=0}^{k} \operatorname{rank} H_n(\Omega(\Re^\ell - X_3)) \times \operatorname{rank} H_{k-n}(\Lambda_{123}) \\ &\geq \operatorname{rank} H_k(\Omega(\Re^\ell - X_3)) \\ &\geq \tfrac{3^r}{r(r-1)}(1 - r3^{-\frac{r}{2}}). \end{aligned} \tag{2.29}$$

But (2.27) implies that

$$\operatorname{rank} H_k(\Lambda) \leq 32 \times 2^r + 8r(\ell-2) \tag{2.30}$$

for r large enough.

(2.29) and (2.30) imply that

$$\frac{3^r}{r(r-1)}(1 - r3^{-\frac{r}{2}}) \leq 32 \times 2^r + 8r(\ell-2) \tag{2.31}$$

and this is false for r large enough.

This contradiction establishes Theorem 2.1.

3. Generalized Morse inequalities for the 4-body problem

Let q be a critical point of I. Let $m(q)$ be the Morse index of q, i.e., $m(q)$ is the maximal dimension of a subspace of $T\Lambda_q$ on which I'' is negative definite. Let $\overline{m}(q)$ be the generalized Morse index of q, i.e., $\overline{m}(q)$ is the maximal dimension of a subspace of $T\Lambda_q$ on which I'' is non-positive. $\overline{m}(q) = m(q)+$ the nullity of I''. The translational symmetry of I implies that

$$\overline{m}(q) - m(q) \geq \ell.$$

Let $\beta_k(\Lambda)$ be the k^{th} Betti number of Λ and let N_k be the number of critical points, q, of I such that $m(q) = k$. Then, we have the following theorem:

Theorem 3.1. *Let V satisfy $(V_1)-(V_6)$. Assume that if $I'(q) = 0$ then $\overline{m}(q) - m(q) = \ell$, i.e., I has only non-degenerate critical points, modulo translations. Then, if $k = r(l-2)$ or $k = r(l-2)+1$, we have*

$$N_k \geq \beta_k(\Lambda) - a\,2^r \geq c\,\frac{3^r}{r(r-1)}$$

for k large enough, where a and c are positive constants and $0 < c < 1$.

Proof. For a fixed k, there exists $M > 0$ such that any homology class $\{z\}$ in $H_k(\Lambda)$ can be represented by a chain z having support in I^M. This follows from the fact that $H_k(\Lambda)$ is finitely generated which is a consequence of the result of E. Fadell and S. Husseini [10]. Let

$$\mathscr{K}_k^M = \{q \in \Lambda \,/\, I'(q) = 0,\ I(q) \leq M \text{ and } m(q) = k\}, \tag{3.1}$$

$$\mathscr{K}^{k-1,M} = \{q \in \Lambda \,/\, I'(q) = 0,\ I(q) \leq M \text{ and } m(q) \leq k-1\}. \tag{3.2}$$

Let

$$A_k = \bigcup_{q \in \mathscr{K}_k^M} W_u(q) \tag{3.3}$$

and

$$B_{k-1} = \bigcup_{q \in \mathscr{K}^{k-1,M}} W_u(q). \tag{3.4}$$

By [19] (Theorem 5.1), the chain z with support in I^M representing $\{z\} \in H_k(\Lambda)$ may be represented as a chain in $H_k(\mathscr{W}^\infty \cup \mathscr{V}_\epsilon(D_{M+1}))$ with $D_{M+1} = \bigcup_{q \in \mathscr{K}^{M+1}} W_u(q)$.

The support of z is provided by a continuous map, which we can approximate by a differentiable map, from the standard k-simplex into I^M. Hence, using a transversality argument, we can assume that the support of z does not meet the stable manifolds, $W_s(q)$, of the critical points q with $m(q) > k$. Therefore, $\{z\}$ may be represented as a chain in $H_k(\mathscr{W}^\infty \cup \mathscr{V}_\epsilon(A_k \cup B_{k-1}))$. Letting $\epsilon \longrightarrow 0$, $\{z\}$ may be represented in $H_k(\mathscr{W}^\infty \cup A_k \cup B_{k-1})$.

We can extend $\mathscr{W}^\infty$ to $\widetilde{\mathscr{W}}^\infty$, a neighborhood of $I^{\epsilon_1} \cup D^\infty$ where D^∞ is the union of the unstable manifolds of all critical points at infinity. Since

$\widetilde{\mathscr{W}}^\infty \cap I^{M+1} = \mathscr{W}^\infty$, each chain of $H_k(\Lambda)$ may be represented as a chain of $H_k(\widetilde{\mathscr{W}}^\infty \cup A_k \cup B_{k-1})$.

Then, we show that

$$H_k(\widetilde{\mathscr{W}}^\infty \cup A_k \cup B_{k-1}, \ \widetilde{\mathscr{W}}^\infty \cup B_{k-1}) = \mathscr{Z}_2^{N_k}. \tag{3.5}$$

Indeed, $A_k \cup B_{k-1}$ fibers locally over B_{k-1} (i.e., there exists a neighborhood U of B_{k-1} such that $(A_k \cup B_{k-1}) \cap U$ is a fiber-bundle over B_{k-1}. This follows from the fact that $W_u(q)$ fibers locally over $W_u(q')$ if $W_s(q') \cap W_u(q) \neq \varnothing$ which is a result proved by A. Bahri and P. Rabinowitz in [6] (Proposition 7.24). On the other hand, $(A_k - \mathscr{K}_k^M) \cup \widetilde{\mathscr{W}}^\infty \cup B_{k-1}$ is invariant under the pseudogradient flow. This implies that $(A_k - \mathscr{K}_k^M) \cup \widetilde{\mathscr{W}}^\infty \cup B_{k-1}$ has the homotopy type of $\widetilde{\mathscr{W}}^\infty \cup B_{k-1}$ since the decreasing flow will bring the complement of $\mathscr{K}_k^M$ in their unstable manifolds (A_k) to neighborhoods of the unstable manifolds of critical points of $\mathscr{K}^{k-1,M}$ or into I^{ϵ_1}. Hence

$$\begin{aligned} &H_k(\widetilde{\mathscr{W}}^\infty \cup A_k \cup B_{k-1}, \widetilde{\mathscr{W}}^\infty \cup B_{k-1}) \\ &\quad = H_k(\widetilde{\mathscr{W}}^\infty \cup A_k \cup B_{k-1}, (A_k - \mathscr{K}_k^M) \cup \widetilde{\mathscr{W}}^\infty \cup B_{k-1}). \end{aligned} \tag{3.6}$$

By excision, we have

$$\begin{aligned} &H_k(\widetilde{\mathscr{W}}^\infty \cup A_k \cup B_{k-1}, (A_k - \mathscr{K}_k^M) \cup \widetilde{\mathscr{W}}^\infty \cup B_{k-1}) \\ &\quad = H_k(A_k, A_k - \mathscr{K}_k^M) = \mathscr{Z}_2^{N_k}. \end{aligned} \tag{3.7}$$

Then (3.5) follows from (3.6) and (3.7).

Since any generator of $H_k(\Lambda)$ can be represented in $H_k(\widetilde{\mathscr{W}}^\infty \cup A_k \cup B_{k-1})$ and this representation is injective, we have

$$\operatorname{rank} H_k(\widetilde{\mathscr{W}}^\infty \cup A_k \cup B_{k-1}) \geq \beta_k(\Lambda). \tag{3.8}$$

Hence, using the exact sequence for the pair $(\widetilde{\mathscr{W}}^\infty \cup A_k \cup B_{k-1}, \ \widetilde{\mathscr{W}}^\infty \cup B_{k-1})$, we get

$$N_k \geq \beta_k(\Lambda) - \operatorname{rank} H_k(\widetilde{\mathscr{W}}^\infty \cup B_{k-1}). \tag{3.9}$$

Following the proof of A. Bahri and P. Rabinowitz [6] for the 3-body problem, we can show, using fibrations, a retraction argument and a Mayer-Vietoris sequence, that

$$\operatorname{rank} H_k(\widetilde{\mathscr{W}}^\infty \cup B_{k-1}) \leq \operatorname{rank} H_k(\widetilde{\mathscr{W}}^\infty). \tag{3.10}$$

(3.10) together with (3.9) imply that

$$N_k \geq \beta_k(\Lambda) - \operatorname{rank} H_k(\widetilde{\mathscr{W}}^\infty). \tag{3.11}$$

Then, by similar arguments to those used in Section 2, if k is large enough, we get

$$\operatorname{rank} H_k(\widetilde{\mathscr{W}}^\infty) \leq 4(\operatorname{rank} H_{k-(l-1)}(\Lambda_{123}) + \operatorname{rank} H_k(\Lambda_{123})) + p(k) \tag{3.12}$$

where $p(k)$ is a polynomial of degree 1 in k.

If $k = r(l-2)$ or $k = r(l-2)+1$, then

$$H_k(\widetilde{\mathscr{W}}^{\infty}) \leq a\,2^r \tag{3.13}$$

where a is a positive constant.

This implies that if $k = r(l-2)$ or $k = r(l-2)+1$, then

$$N_k \geq \beta_k(\Lambda) - a\,2^r \geq \frac{3^r}{r(r-1)}(1 - r3^{-\frac{r}{2}}) - a\,2^r \geq c\,\frac{3^r}{r(r-1)} \tag{3.14}$$

where c is a positive constant, $0 < c < 1$.

4. Generalization to the n-body problem

In this section, $n \geq 4$ and

$$V = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} V_{ij}(t, q_i - q_j)$$

where the functions V_{ij} satisfy $(V_1) - (V_6)$.

We consider the Hamiltonian system

$$\text{(HS)} \qquad \begin{cases} \ddot{q} + V'(q) = 0 \\ q(t+T) = q(t), \ \forall t \in \Re. \end{cases}$$

Posing (HS) as a variational problem, we look for critical points of the functional

$$I(q) = \int_0^T (\frac{1}{2}|\dot{q}|^2 - V(q))\,dt.$$

The main result is:

Theorem 1. *If V satisfies $(V_1) - (V_6)$, then for each $T > 0$, I possesses an unbounded sequence of critical values.*

To prove Theorem 1, we assume that the set of critical points of I is bounded by a. We choose $M > a$, then using the retraction result of [19], together with appropriate Mayer-Vietoris sequences, we get the following inequality for k large enough:

$$\operatorname{rank} H_k(\Lambda_n) \leq \sum_{\substack{p_1+\dots+p_r\leq k \\ n_1+\dots+n_r\leq n \\ 2\leq n_i\leq n-1}} c(n_1, \dots, n_r)\operatorname{rank} H_{p_1}(\Lambda_{n_1})\dots\operatorname{rank} H_{p_r}(\Lambda_{n_r})$$

where $c(n_1, \dots, n_r)$ are positive constants. These constants take into account the interaction between the different packs defining the critical points at infinity and are given by the homology of sets having the homotopy type of a sphere or a wedge of spheres in $\Re^\ell$.

On the other hand, by a result of E. Fadell and S. Husseini [10], using coefficients in $\mathscr{Z}_2$, we have

$$H_*(\Lambda_n) = H_*(\Omega(\Re^\ell - X_{n-1})) \otimes \dots \otimes H_*(\Re^\ell - X_1)$$

where $X_j = \{x_1, x_2, \dots, x_j\}$ and $x_r \neq x_s$ for $r \neq s$.

We also know by results of M. Vigué-Poirrier [20] and E. Fadell and S. Husseini [10], that

$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j); \mathscr{Z}_2) \neq 0$ only if $k \equiv 0, 1 \pmod{\ell - 2}$,

$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j); \mathscr{Z}_2) \leq j^r$ for $k = r(\ell - 2)$ and $k = r(\ell - 2) + 1$

and

$\operatorname{rank} H_k(\Omega(\Re^\ell - X_j); \mathscr{Z}_2) \geq \frac{j^r}{r(r-1)}(1 - rj^{-\frac{\ell}{2}})$ for $k = r(\ell - 2)$ and $k = r(\ell - 2) + 1$.

Hence, if the set of critical values of I is bounded then

$\operatorname{rank} H_k(\Lambda_n) \leq f_n(k)(n-2)^r$ if $k = r(l-2) + s$, $0 \leq s < l - 2$ where $f_n(k)$ is a polynomial in k with coefficients depending on n.

This gives us a contradiction, since

$$\begin{aligned} \operatorname{rank} H_k(\Lambda_n) &\geq \operatorname{rank} H_k(\Omega(\Re^\ell - X_{n-1})) \\ &\geq \frac{(n-1)^r}{r(r-1)}(1 - r(n-1)^{-\frac{\ell}{2}}) \end{aligned}$$

for $k = r(\ell - 2)$ and $k = r(\ell - 2) + 1$.

Finally, when I has only non-degenerate critical points, modulo translations, i.e., $\overline{m}(q) - m(q) = \ell$ for any critical point q where $m(q)$ is the Morse index of q (number of negative eigenvalues of $I''(q)$ counting multiplicities) and $\overline{m}(q)$ is the generalized Morse index of q (number of negative or zero eigenvalues of $I''(q)$), we can show using the retraction argument of [19] that, if N_k denotes the number of critical points, q, of I with Morse index $m(q) = k$ then

$$N_k \geq \operatorname{rank} H_k(\Lambda_n) - f_n(k)(n-2)^r \geq c_n \frac{(n-1)^r}{r(r-1)}$$

for $k = r(\ell - 2)$ and $k = r(\ell - 2) + 1$ if r is large enough. Here, c_n is a positive constant which depends on n and satisfies $0 < c_n < 1$.

If V is autonomous, the requirement $\overline{m}(q) - m(q) = \ell$ cannot be satisfied. Indeed, the resulting S^1 invariance of I implies that critical points occur in circles. Therefore, $\overline{m}(q) - m(q) \geq \ell + 1$ for any critical point q of I.

In this case, following Bahri-Rabinowitz [6], we define $\overline{N}_k$ as the number of critical circles of Morse index k, and assume that $\overline{m}(q) - m(q) = \ell + 1$ for any critical point q of I. Then,

$$\overline{N}_k + \overline{N}_{k-1} \geq c_n \frac{(n-1)^r}{r(r-1)}$$

for $k = r(\ell - 2)$ and $k = r(\ell - 2) + 1$.

To show this, I is perturbed slightly so that any circle of critical points is broken up into two critical points, one minimum and one maximum of the functional on the circle. Hence, one critical point has Morse index k and the other has Morse index $k + 1$.

Acknowledgments

This work was done as part of the author's Ph.D. thesis under the direction of Professor Abbes Bahri at Rutgers University, New Jersey. The author is very grateful to Professor Abbes Bahri for the useful discussions and for the support and encouragement, and thanks E. Fadell and S. Husseini for providing her with the topological results needed to complete this work. The author also thanks the members of the Mathematics Department at Rutgers University for their help and would like to acknowledge the hospitality of the International Center for Theoretical Physics, Trieste, Italy, where this paper was completed.

REFERENCES

1. A. Ambrosetti and V. Coti Zelati, *Critical points with lack of compactness and applications to singular dynamical systems*, Ann. Mat. Pura Appl. (4) **149** (1987), 237–259.
2. ______, *Perturbation of Hamiltonian systems with Keplerian potentials*, Math. Z. **201** (1989), 227–242.
3. A. Bahri, Thèse de Doctorat d'Etat, Univ. P. et M. Curie, Paris, 1981.
4. A. Bahri and H. Berestycki, *Forced vibrations of superquadratic Hamiltonian systems*, Acta Math. **152** (1984), 143–197.
5. A. Bahri and P.H. Rabinowitz, *A minimax method for a class of Hamiltonian systems with singular potentials*, J. Funct. Anal. **82** (1989), 412–428.
6. ______, *Periodic solutions of Hamiltonian systems of 3-body type*, Ann. Inst. H. Poincaré, Analyse, Non Lineaire **8** (1991), 561–649.
7. V. Coti Zelati, *Periodic solutions for N-body type problems*, Ann. Inst. H. Poincare, Analyse Non Linéaire **7** (1990), 477–492.
8. M. Degiovanni and F. Giannoni, *Dynamical systems with Newtonian type potentials*, Ann. Sci. Norm. Sup. Pisa **15** (1988), 467–494.
9. A. Dold, *Lectures on algebraic topology*, Springer-Verlag, 1980.
10. E. Fadell and S. Husseini, *On the growth properties of the homology of free loop spaces on configuration spaces*, Proc. Conf. on Non Linear Analysis, Florida, 1992.
11. W.B. Gordon, *Conservative dynamical systems involving strong forces*, Trans. Amer. Math. Soc. **204** (1975), 113–135.
12. C. Greco, *Periodic solutions of a class of singular Hamiltonian systems*, Nonlinear Analysis, T.M.A. **12** (1988), 259–270.
13. O. Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Mat. **1** (1952), 389–408.
14. M.W. Hirsch, *Differential topology*, Springer-Verlag, 1988.
15. A. Marino and G. Prodi, *Metodi perturbativi nella teoria di Morse*, Boll. Un. Mat. Ital. **11** (1975), 1–32.
16. P. Majer and S. Terracini, *Periodic solutions to some problems of n-body type*, Arch. Rational Mech. Anal. **124** (1993), 381–404.
17. ______, *Multiple periodic solutions to some n-body type problems via a collision index*, to appear.
18. H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Libr. Albert Blanchard, Paris, 1987.
19. H. Riahi, *Study of the critical points at infinity arising from the failure of the Palais-Smale condition for n-body type problems*, Mem. Amer. Math. Soc. (to appear).
20. M. Vigué-Poirrier, *Homotopie rationnelle et croissance du nombre de géodésiques fermées*, Ann. Sci. Ecole Norm. Sup. **17** (1984), 413–431.

DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES DE TUNIS, CAMPUS UNIVERSITAIRE, LE BELVEDERE, TUNIS, TUNISIA

Current address: Ecole Nationale d'Ingénieurs de Tunis, Laboratoire de Mathématiques Appliquées, BP 37 Campus Universitaire, Le Belvédère, 1060, Tunis, Tunisia

E-mail address: `riahih@stardent.cck.tn`